The Power of the Continuum.

Inaugural-Dissertation

zur

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Von

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Dedicated

to

Mrs. Pittard-Bullock.

Sources.

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2) G. Cantor, Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. Crelle's Journal 77, page 258.
3) Math. Annalen 4, page 4,
4) Per. di mat. 6, 1891, page 14, Betazzi.
5) Jahrbuch f. Math. 76, du Bois-Reymond.
6) Math. Annalen 21, page 545.
7) E. Zermelo, Math. Annalen 59, 4, page 56.
8) Felix Bernstein, Untersuchungen aus der Mengenlehre, 1901, page 49.
9) Bull. de la Soc. de Fr. 84.
10) Jahresbericht d. d. Mathvgg. I, page 77.
11) Journal f. Math. 84, page 243.
12) Dito, page 256, — for the multitude of all functions of a real variable.
13) cf. 8), § 6.
14) Jahresbericht d. d. Mathvgg., Vol. 5, page 81, Article by Herr Schröder.

Preface.

The purpose of this thesis, to furnish the proof that the power of the continuum is the lowest but one, or in other words, to prove that there is no multitude the power of which is lower than that of the continuum and at the same time higher than that of a dinumerable multitude, necessitates the development of a number of theorems with a view to their application.

They are for the greater part due to authorities on the theory of multitudes and are widely known, so that in many cases a few short remarks on the principles of the proof in question appear to be sufficient. On the other hand it is hardly possible to avoid these notes, as in the theory of multitudes special care should at present be taken against the possibilities of the circulus.

This essay — an abstract of part of an extensive Theory of Multitudes in preparation — was occasioned by a course of lectures delivered by Dr. Edmund Landau at the Friedrich Wilhelms-Universität, Berlin. The writer is furthermore indebted to Prof. Georg Cantor, Ph. D. etc., Halle-Saale, through whom in former years he became acquainted with the discipline treating multitudes and sets of points.

Charlottenburg-Berlin, Jan. 13. 1905.

Historical Introduction.

History shows that every discovery — like every invention of importance — has numberless predecessors which, accumulating and complementing each other, prepare the ground on which the res nova then appears as natural and necessary consequence of a continuous sequence of details — though to human shortsightedness a certain degree of incontinuous abruptness seems to characterize the event. It has for instance been pointed out, that Newton and Leibnitz are not the actual founders of the infinitesimal calculus, their real merit being the introduction and deepening of the theory of functions of a real variable as this theory.*)

Similar observations may be made regarding the theory of multitudes, although here earlier signs are difficult to trace.

Herr Georg Cantor is looked upon as the discoverer and creator, and in rare cases has a discovery been attributed to one man alone with more readiness. He is the first who, in investigating more or less curious methods of concluding, in observing facts of various mathematical disciplines, and finally in searching for

*) An earlier author on Differential Calculus is Joh. Hudde. cf. De Maximis et Minimis 1659, cf. also Encyklopädie d. Math. Wissenschaften, Vol. II A. 2. (A. Voss) and Moritz Cantor, Vorlesungen üb. d. Geschichte d. Mathematik.

non-algebraical numbers, recognised the roots of conclusions capable of far reaching deductions, which have grown to an independant discipline in less than a generation.

It would lead a little far to dwell minutely on cases in which mathematicians of various epochs and in the different branches of mathematical science, have approached the methods and means of the present theory of multitudes; likewise it would hardly be of interest to enter into details on those cases, in which theorems belonging originally to some other discipline are dealt with in the simplest manner conceiveable by application of the theory of multitudes. As an instance however, the proof of the existance of transcendental numbers may follow:

1.) The multitude of all real numbers, the linear continuum, is non-dinumerable.

2.) The multitude of algebraical numbers is dinumerable.

.: the continuum contains elements which are not algebraical. q. e. d.

This reminds one of an opinion frequently expressed, viz. that a mathematical fact is capable of but one proof; should this thesis be correct, then the question of the simplest proof of a theorem would be reduced to the problem of finding the most far reaching theorem — of any discipline — rendering the sought-for theorem as one of its direct corollaries. The opinion mentioned above is evidently based on a conception of taking the sequence of logical conclusions to be a chain without branches, whereas a co-ordination in a higher dimensional space of several immediate deductions from a single combination of facts, appears

to allow of an unconstrained explanation of co-existing proofs to the same theorem.

Cantor arrived at the idea of dinumerability in 1873[1]) and shortly after propounded the theorem that the n-dimensional continuum is of the same power as the linear. A short essay, printed in Vol. 77 of Crelle's Journal, proves the dinumerability of the algebraical numbers, and that an unlimited amount of non-algebraical numbers may be inserted between two elements of the mentioned dinumerable multitude.*) This is the generalisation of Minnigerode whose theorem, aided by Galois' ideas, treated a special case in which the multitude under discussion is finite and the degree of the rational function in question is given [2]) [3]).

During the next two decades, one after the other theorems of more and more general qualities were published, the most characteristic ones for the development of the theory of multitudes being the following:

In 1873 the first proof for the non-dinumerability of the continuum was published, the method of conclusion being based on the idea, that elements defined by a processus in infinitum prevent the multitude in question being dinumerable if these elements have no finite indices in the multitude when postulated as dinumerable.**)

The introduction of transfinite numbers, that is of symbols for derivates of sets of points, arithmetically defined and subject to the laws of algebra,

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*) Before this Liouville had shown that there is an unlimited quantity of non-algebraical, that is transcendental numbers in any intervall $\alpha \ldots . \beta$[2].

**) cf. Betazzi[4] and du Bois-Reymond.[5]

took place in 1882. In the same year a highly important step was taken in the foundation of the theory of multitudes, the introduction of eutaxitic multitudes.[6])

The definition*) of a eutaxitic multitude purports that the multitude itself as well as every submultitude possesses an earliest element. The value therefore lies chiefly in the fact that every sequence of precedence

$$a_1 \succ a_2 \succ a_3 \succ a_4 \; . \; . \; . \; .$$

breaks off.**) The introduction of eutaxy gave rise to numerous gereralising publications, which developed into a system of theorems, by the aid of which Herr Zermelo[7]) of Göttingen in his latest essay has solved the problem of the comparableness of two multitudes. The problem originally deals with the possibility of eutaxy of the continuum, and Zermelo proves the still more far reaching theorem, that every multitude is capable of eutaxy.

At more or less extended intervals after 1882 follow the publication on multitudes of higher power, on the general arithmetics of ordinalia and their normal form, then on general theorems dealing with sets of points, and finally on complete, separated, dense and perfect multitudes. Strong influence was hereby acquired on the theory of functions, as at this point a reciprocal correlation with various disciplines was established.***)

On page 30 of the „Grundlagen einer allgemeinen

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*) The definition given in Vol. 49, page 207, of the Math. Annalen differs but in the words used in Vol. 21, page 545.

**) $a_1 \succ a_2$ signifies a_1 „posterior to“ a_2.

***) cf. also Encyklopädie d. Math. Wissenschaften, Part I, A. 5. article by Schoenflies, also Jahresbericht der deutschen Mathematiker-Vereinigung, Vol. 8, 2, report by Schoenflies.

Mannigfaltigkeitslehre, Ein Mathematisch - Philosophischer Versuch in der Lehre des Unendlichen“ Cantor has the following lines:*)

„Es reduciert sich (daher) die Untersuchung und Feststellung der Mächtigkeit von G_n **) auf dieselbe Frage***), specialisiert auf das Intervall 0 . . . . 1 und ich hoffe, sie schon bald durch einen strengen Beweis dahin beantworten zu können, dass die gesuchte Mächtigkeit keine andere ist als diejenige unserer zweiten Zahlenklasse.“

None of Cantor's later works have thrown light on the position of the power of the continuum, though there is no great difficulty in following traces tending towards this gaol through nearly all his more recent publications. The conjecture, that there is no link between the power of dinumerable multitudes and that of the continuum, has found a deal of both sympathy and contention amoug authorities, no proof however has been published so far for either one or the other view.

At the 3. International Congress of Mathematicians, Heidelberg, in August 1904, Herr König, Budapest, read a paper proving that the continuum is not capable of eutaxy. In the course of his proof he applied the following theorem from Herr Felix Bernstein's Inaugural-Dissertation[8]):

„Between the cardinalia $\aleph_\mu$ and $\aleph_\nu$ of any two eutaxitic multitudes the following equation is established:

$$\aleph_\mu^{\aleph_\nu} = 2^{\aleph_\nu} \aleph_\mu$$

*) 1883. B. G. Teubner.

**) n-dimensional „space“.

***) one-dimensional continuum.

Before the congress closed however Cantor succeeded in showing that this theorem, though correct in the special case to which Bernstein had applied it, contained an error in its generalized form, and that therefore König's proof, which was instantly withdrawn, was founded on an incorrect basis.

Had König been right, the problem of the power of the continuum would also have been settled without delay, though not according to Cantor's expectations, that is the power of the continuum would have been found not to be the lowest but one. — Neither the failure of Königs proof, nor Zermelo's proof to the contrary have cleared the situation up.*) — The writer believes to have demonstrated on the following pages that Cantor's conjectures regarding the position of the power of the continuum are correct. When he heard of the problem first, knowledge of positive qualities of transcendental numbers appeared to him to be the inevitable means of investigation.

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*) Tannery's endeavours can not be pronounced complete.

Special Theorems.

Definition. A system of an infinite number of objects, for instance of mathematical quantities be called a multitude. A system of an infinite number of numbers, for instance of all real positive intergers, be called a multitude of points, or set of points, though in the following chapters we will use the simpler term multitude alone, as none but those of points occur.

Notice should be taken of the adviseability of retaining the term multitude, even in those special cases in which the number of elements sinks to a finite quantity.

A multitude is called dinumerable if a sequence of indices can be made to correspond to its elements in such a manner that the position of the nth, $(n + 1)$ th . . . . element can be located. Evidently the multitude of all prime numbers, of the squares of prime numbers, the multitude of all positive even numbers etc. are dinumerable multitudes, as arranging the elements according to their value and supplying them with consequent indices will fulfil the definition of dinumerability. The multitude of all real intergers and that of all positive rational fractions below 1 contains neither a greatest nor smallest element, nevertheless a glance at the following two tables show that they are dinumerable multitudes:

I.	.... -4	-3	-2	-1	0	1	2	3	4 ....
Elements:	= 0	1	-1	2	-2	3	-3	4	
Indices:	1	2	3	4	5	6	7	8	

II.

Elements:	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\left(\frac{2}{4}\right)$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	....
Indices:	1	2	3	4		5	6	7	

Further investigation of various multitudes would easily lead to a dinumerable arrangement, so that we may incline to expect all multitudes to be dinumerable and the possible difficulty to lie merely in successful search for a fitting arrangement. — However, non-dinumerable multitudes do exist, one of which, here demanding our special interest, being the multitude of all numbers, the continuum.

The power of the n-dimensional continuum C_n is equal to the power of the linear C_1, i. e. the various continua are equivalent, so that the following relation takes place:

$$C_n \sim C_1$$

The proof appears simplest by demonstrating the equivalence of all numbers of a plane to those of the real axis:

By central projection, the illimited plane may be depicted on the face of, for instance, a square. We will not take the rational numbers into consideration as they form dinumerable multitudes and are therefore equivalent. The elements remaining are irrational. An irrational number (x, y) of the plane may be written in the following form:

$$x = \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cfrac{1}{\beta_3 + \cfrac{1}{\beta_4 + \cdots}}}}$$

$$y = \cfrac{1}{\gamma_1 + \cfrac{1}{\gamma_2 + \cfrac{1}{\gamma_3 + \cfrac{1}{\gamma_4 + \cdots}}}}$$

so that the corresponding number z of the real axis is:

$$z = \cfrac{1}{\beta_1 + \cfrac{1}{\gamma_1 + \cfrac{1}{\beta_2 + \cfrac{1}{\gamma_2 + \cdots}}}}$$

The manner of correspondency between the elements of the n-dimensional and the linear continuum is analoguous.

If $(x_1, x_2, x_3, x_4, \ldots . x_n)$

denote the symbol of an irrational quantity in the n-dimensional space, the following system of expressions will be found:

$$x_1 = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_{n+1} + \cfrac{1}{\alpha_{2n+1} + \ddots}}}$$

$$x_2 = \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_{n+2} + \cfrac{1}{\alpha_{2n+2} + \ddots}}}$$

$$\vdots$$

$$x_n = \cfrac{1}{\alpha_n + \cfrac{1}{\alpha_{2n} + \cfrac{1}{\alpha_{3n} + \cfrac{1}{\alpha_{4n} + \ddots}}}}$$

so that the corresponding element x of the real axis appears in the following form:

$$x = \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \ddots + \cfrac{1}{\alpha_n + \cfrac{1}{\alpha_{n+1} + \cfrac{1}{\alpha_{n+2} + \ddots +}}}}}$$

$$\cfrac{1}{\alpha_{2n} + \cfrac{1}{\alpha_{2n+1} + \cdots}}$$

In proving the continuum non-dinumerable we will take the theorem for granted that every x satisfying

$$0 \leqq x < 1$$

can be written as a decimal fraction, all figures α_ν of which satisfy

$$0 \leqq \alpha_\nu < 9$$

The proof itself is the following:

Supposing the part of the continuum between 0 and 1 (excluding both) were dinumerable, i. e. could be arranged for instance as follows:

$$c_1 \; c_2 \; c_3 \; c_4 \; \ldots .$$

then according to the postulated theorem the following system of equations could be constructed:

$$\begin{array}{l} c_1 = 0,\ \alpha_{1,1}\ \alpha_{1,2}\ \alpha_{1,3}\ \alpha_{1,4}\ \ldots . \\ c_2 = 0,\ \alpha_{2,1}\ \alpha_{2,2}\ \alpha_{2,3}\ \alpha_{2,4}\ \ldots . \\ \vdots \\ c_n = 0,\ \alpha_{n,1}\ \alpha_{n,2}\ \alpha_{n,3}\ \alpha_{n,4}\ \ldots . \\ \vdots \end{array}$$

where the expressions to the right of the signs of equality are symbolic terms for decimal fractions. This arrangement is always short of a certain multitude of numbers, although they belong to the continuum. The elements of this missing multitude are numbers t_ν, differing when written as decimal fractions, in one place from each term c_ν, for instance:

$$t_1 = 0, \beta_1\ \beta_2\ \beta_3\ \beta_4 \ldots .$$

$\beta_1 \ldots . \beta_4 \ldots .$ satisfying the following double system of inequalities:

$$\beta_1 \lessgtr 9 \text{ and } \beta_1 \lessgtr \alpha_{1,1}$$

$$\beta_2 \lessgtr 9 \quad \text{„} \quad \beta_2 \lessgtr \alpha_{2,2}$$

$$\vdots$$

$$\beta_n \lessgtr 9 \quad \text{„} \quad \beta_n \lessgtr \alpha_{n,n}$$

$$\vdots$$

So the part between 0 and 1 of the continuum is not dinumerable (cf. also introductory page 11), and therefore the unlimited continuum itself a fortiori also not*), like the non-dinumerability of any portion of the continuum may be demonstrated independently by the same method.[10])

In combining observations on certain dinumerable

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*) Strictly the portion between 0 and 1 is equivalent to the whole between $-\infty$ and $+\infty$.

multitudes with this inquiry into some of the qualities of the continuum, the question arises as to which elements originate the non-dinumerability of the continuum. In order to arrive at the solution of this question the outlines of Cantor's proof for the dinumerability of the multitude of abgebraical numbers must follow:

D e f i n i t i o n. The sum of the absolute values — i. e. regardless of their signs — of the coëfficients plus the degree of the equation, is called the a l t i t u d e of the expression; for instance the altitude H of the algebraical equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-3} + \ldots\ldots + a_n = 0$$

is given by

$$H = n + |a_0| + |a_1| + \ldots\ldots + |a_n|$$

This definition of the altitude is the more adviseable one for the case in question, as evidently an unlimited number of equations would correspond to the same altitude, if defined by the sum of the coëfficients only.

Taking a_0 not to be zero and a_0, a_1, $a_2 \ldots . a_n$ to have no more common factors we proceed to construct consecutively equations characterized by their increase of altitude, the radicals of which are the potential algebraical numbers. We therefore arrange the multitude of algebraical numbers according to the increasing altitudes of their original equations and within a group of radicals according to their value. The first few steps of this arrangement given below will serve to prevent any mistake:

Equations of the altitude 1 do not exist.

The only equation of the altitude 2 is

$$1 \,.\, x^1 = 0$$

0 therefore is the first algebraical number.

For $H = 3$ we find:

1.) $n = 1$ and $\Sigma \mid a_\nu \mid = 2$

2.) $n = 2$ and $\Sigma \mid a_\nu \mid = 1$

where n is to signify the degree, a_ν the coëfficients. The equations corresponding to $H = 3$ are:

$$x + 1 = 0$$
$$x - 1 = 0$$
$$x^2 = 0$$
$$2\,x = 0$$

rendering $- 1$ and $+ 1$ as the two following elements of the multitude of algebraical numbers.

$H = 4$ allows of the following three possibilities:

1.) $n = 1$ and $\Sigma \mid a_\nu \mid = 3$

2.) $n = 2$ and $\Sigma \mid a_\nu \mid = 2$

3.) $n = 3$ and $\Sigma \mid a_\nu \mid = 1$

resulting in the following system of equations:

$$x + 2 = 0$$
$$x - 2 = 0$$
$$2\,x + 1 = 0$$
$$2\,x - 1 = 0$$
$$x^2 + x = 0$$
$$x^2 - x = 0$$
$$x^2 + 1 = 0$$
$$x^2 - 1 = 0$$
$$3\,x = 0$$
$$x^3 = 0$$

with the following radicals:

$$
\begin{aligned}
x &= -2 \\
x &= 2 \\
x &= -\tfrac{1}{2} \\
x &= \tfrac{1}{2} \\
x &= 0 \text{ and } -1 \\
x &= 0 \quad " \quad +1 \\
x &= i \quad " \quad -i \\
x &= 1 \quad " \quad -1 \\
x &= 0 \\
x &= 0
\end{aligned}
$$

The newly found elements are -2, $+2$, $-\frac{1}{2}$ $+\frac{1}{2}$ with the corresponding indices 4 to 7, so that the first part of the arranged multitude of algebraical numbers offers an aspect as follows:

Elements:	0	-1	$+1$	-2	$-\frac{1}{2}$	$+\frac{1}{2}$	$+2$	....
Indices:	1	2	3	4	5	6	7	

Herewith the question after the elements effecting the non-dinumerability of the continuum is also answered, as the multitude of all real numbers consisting of the multitude of algebraical and that of transcendental numbers, and the former being dinumerable, it follows with conclusive evidence that the transcendental numbers are the elements preventing the continuum being dinumerable.

Further deductions show that the multitude of transcendental numbers is itself not dinumerable, as, postulating it were dinumerable, i. e. could be arranged in the following manner:

$$T = t_1 \; t_2 \; t_3 \; t_4 \; \ldots .$$

then the sum of the multitudes of transcendental and alge-

braical numbers could be written in the following form :*)

$$\text{Elements: } C = A + T = \begin{array}{cccccccccc} 0 & -1 & +1 & -2 & -\frac{1}{2} & & & & & \\ & & & & \ldots & t_1 & t_2 & t_3 & t_4 & \ldots \end{array}$$

$$\text{Elements: } = \left| \begin{array}{cccccc} 0 & -1 & +1 & -2 & -\frac{1}{2} & \ldots \\ t_1 & t_2 & t_3 & t_4 & t_5 & \ldots \end{array} \right.$$

$$\begin{array}{llcccccccc} \text{Elements:} & = & 0 \; t_1 & - & 1 \; t_2 & + & 1 \; t_3 & - & 2 \; t_4 & \ldots \\ \text{Indices:} & & 1 \; 2 & & 3 \; 4 & & 5 \; 6 & & 7 \; 8 & \end{array}$$

thus C, the continuum, would appear dinumerable, so the assumption is incorrect, and the multitude of transcendental numbers is not dinumerable.

The above operation, cumulative addition of two multitudes, is capable of generalization, thus giving rise to the following theorem:

„A dinumerable multitude of dinumerable multitudes is dinumerable“[11])

According to the premise Π_m satisfies the following symbolic equation:

$$\Pi_m = P_{m_1} \; P_{m_2} \; P_{m_3} \; P_{m_4} \; \ldots$$

furthermore P_{m_1} P_{m_2} .... may be written thus:

$$\begin{array}{lcccccc} P_{m_1} & = & a_1 & b_1 & c_1 & d_1 & \ldots \\ P_{m_2} & = & a_2 & b_2 & c_2 & d_2 & \ldots \\ P_{m_3} & = & a_3 & b_3 & c_3 & d_3 & \ldots \\ P_{m_4} & = & a_4 & b_4 & c_4 & d_4 & \ldots \\ \vdots & & & & & & \end{array}$$

*) The term sum is to denote a cumulation of the elements of either group without actual addition. Evidently it is not adviseable to apply the term when the multitudes in question have elements in common.

A glance at this table shows that a dinumerable arrangement is acquired by counting the elements successively along the inclined arrows, so that

Elements:	$\Pi_m =$	a_1	a_2	b_1	a_3	b_2	c_1	a_4	b_3	c_2	d_1	....
Indices:		1	2	3	4	5	6	7	8	9	10	

is the dinumearble result of the cumulation mentioned. The conclusion arrived at above, viz: that the multitude of transcendental numbers is not dinumerable, enables us by the following reflections to deduct that this multitude is equivalent to the continuum, or, expressed in more general terms, that a non-dinumerable multitude remains non-dinumerable and equivatent to itself if a dinumerable multitude has heen dissociated i. e. every element of the main multitude will correspond to an element of the remaining submultitude in a reversable monological manner, in figures:

$$U - A \smile U$$

where U is to denote a non- dinumerable and A a dinumerable multitude, the sign of minus to be taken analognous to the plus explained on page 24.

$$U - A \smile U$$

is identical with

$$U - A = U'$$

U' representing a submultitude of U.

1.) $$\therefore U = U' + A$$

It is always possible to detach a dinumerable multitude from any given one, so we are justified in formulising

2.) $$U' = U'' + A'$$

U' denoting a not necessarily non-dinumerable and

A′ a dinumerable submultitude of U′. 1.) and 2.) combined render

$$U = U'' + A' + A$$

and as two countable multitudes remain countable when cumulated, we get

3.) $$U = U'' + A''$$

The right hand parts of equations 2) and 3) are equivalent, as the non-dinumerable portions are identical and two diunmerable infinities always allow of a reversable monological correspondency; therefore the left hand parts also must be equivalent, i. e. we find

$$U \sim U'$$

or $$U \sim U - A \text{ q. e. d.}$$

The proof of the theorem put forward at the end of the introduction is partly based on a reflection that multitudes of lines, the lengths of which approach the lower limit zero (excl), and which are found by imagining an infinite straight line divided ad libitum, though with the restriction, that two particles have no more than one point in common, is dinumerable.

The truth of this becomes evident by dividing the original straight line into a countable multitude of finite parts, and then within each of these parts arranging those particles whose lenghts l satisfy

$$l \geqq \vartheta$$

where $O < \vartheta < 1$; the number of these lines is finite inside one of the larger finite parts, therefore dinumerable. After this we proceed to arrange according to

$$\vartheta > l \geqq \vartheta^2$$

and so forth, the result being a dinumerable multitude of dinumerable multitudes of finite multitudes, i. e. a dinumerable multitude.

The Power of the Continuum.

If A represent a dinumerable multitude it is not equivalent to the continuum C, inasmuch as the latter is of higher power than the former.

The non-dinumerability of C is due to the transcendental elements as the multitude of algebraical numbers is dinumerable — page 21 —. Further as a non-dinumerable multitude diminished by a dinumerable one remains non-countable, equivalent to itself — page 25 — the multitude T of transcendental numbers is non-dinumerable and equivalent to C.

Cumulation of dinumerable submultitudes of T to A would result in ad dinumerable multitude — page 24 —, addition of non- dinumerable submultitudes would result in a non-dinumerable multitude of at least the power of C. The third possibility, addition of a non - dinumerable multitude of dinumerable multitudes would likewise result in a multitude equivalent to C, or of higher power, as the multitude P of points on a line, or of lines, L, the lengths of which approach the lower limit zero (excl.) are dinumerable. — page 26 — Assuming P to be non-dinumerable, the non-dinumerability of the first elements of the elements of P would follow, and thus the non-dinumerability of A enlarged by P.

As the idea of the multitude of transcendental

numbers excludes other possibilities of division than those given,*) we arrive at the following

Theorem. Every multitude of lower power than that of the n- dimensional continuum ($n > o$)**) is dinumerable.

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*) cf. Transcendental Numbers in Theory of Multitudes (in preparation), also page 23.

**) For n denoting non-dinumerable infinities cf. Theory of Multitudes.

Eutaxy of the Continuum.

It is evident from the last chapter that any multitude of points is either diunmerable or equivalent to the continuum, as the idea of multitudes of points beyond the power of the continuum includes a contradiction of itself. However, multitudes in general of a higher power may be constructed by forming the multitude of all submultitudes of the continuum, then by forming the multitude of all submultitudes of the last constructed etc.

Let U denote the multitude of all submultitudes of C, then the proof of

$$U > C$$

is the following: [12])

The proposition $U > C$ is identical with

$$\left|\begin{array}{ll} U_1 \sim C & \quad U_1 \text{ denoting a submultitude of } U \\ U \not\sim C & \end{array}\right.$$

There is no difficulty whatever in disengaging a submultitude $U_1 \sim C$ from C, so the problem is reduced to proving

$$U \not\sim C$$

Here two cases are possible, the first of which, viz:

$$U \sim A$$

A, being dinumerable, is discharged by the following observations:

Assuming

$$U \frown A$$

a submultitude U_1 could be found, so that

$$U_1 \frown A$$

were correct, and where U_1 might denote any submultitude. As however according to the obove remark a U_1 does exist for which

$$U_1 \frown C$$

is correct, we would necessarily have to conclude that

$$C \frown A$$

be correct, which would be a contradiction to the non-dinumerability of the continuum.

We proceed to demonstrate the contradiction contained in

$$U \frown C$$

Supposing C were equivalent to U, then every element x of C would correspond to an element M_x of U, where M_x is to denote the submultitudes of C, i. e. elements of U. We divide all elements x into two classes, the first of which is to contain all those which at the same time are elements of M_x, while the second is to contain all others. The elements of the first class form a multitude D, those of the second a multitude B, so that

$$D + B = C$$

We construct a multitude y_B equivalent to B,

$$\text{I.)}\ y_B \frown B$$

the elements of which we split into two classes, the same as above. In considering any one of the elements y the following two cases may occur:

1.) y belongs to the first class, i. e. to D, therefore in consequence of the equivalence I.) also to B, which means that y is an element of the second class also. 2.) y belongs to the second class, i. e. to B, and therefore according to 1.) to the first class also as it occurs in $y_B \frown B$.

Therefore all elements would belong to both classes, which is not possible as the two classes are complementary ($D + B = C$), and the actual existence of both classes is beyond doubt. The proposition therefore is correct.

The Power of W.

The following considerations are to show that the multitude W of types is also of the lowest power but one[13]) and therefore equivalent to the continuum.

Definition. Two equivalent multitudes shall be called homologically arranged, when they are arranged in such a manner that, two elements m and n of the first multitude satisfying the condition

$$m \prec n$$

the corresponding elements p and q of the second one satisfy.

$$p \prec q$$

A law or rule of arrangement shall be termed type of arrangement. Cantor's definition is the following:

„Der Ordnungstypus oder Typus von M ist der Allgemeinbegriff, der sich ergiebt, wenn man von der

Beschaffenheit der Elemente abstrahiert, die Rangordnung unter ihnen aber beibehält."

If W denote the multitude of types, homologically arranged multitudes being represented by one type, and if further

*) 1.) M be dinumerable and eutaxitic
2.) N " " " "
3.) $M \mathrel{\not\equiv} N$

where M and N are to denote types and the symbol in 3) is to express their dissimilarity, then

either $M \equiv N_1$ — N_1 denoting a section of M
or $M_1 \equiv N$ — M_1 " " " " N

takes place.

M is termed prior to N. $M \prec N$, if $M \equiv N_1$
In the same way we have

$$N \prec M$$
$$\text{for } N \equiv M_1$$

Let a, b, c be types corresponding to M_1 N_1 P and let

$$a \prec b$$
$$\text{and } b \prec c$$

take place, then M will be a section of N, and N of P, formulatory:

$$M \mid N$$
$$\text{and } N \mid P$$

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*) Theorem of Equivalence, Cantor [14].

These two expressions render

$$M \mid P$$

and consequently

$$a \prec c$$

The meaning of this is, that three elements of W subject to $a \prec b$ and $b \prec c$ must also obey $a \prec c$, that is: W is simply arranged. The proof of the eutaxy of W is the following.*)

Definition. A multitude is eutaxitic, if every submultitude as well as the multitude itself possess a first element.

Assuming a submultitude of W to possess no first element and a_1 to be any one of its elements, then an element a_2 is sure to exist satisfying

$$a_2 \prec a_1$$

$$\therefore \text{ also } a_3 \prec a_2 \prec a_1$$

$$\therefore \quad a_1 \succ a_2 \succ a_3 \succ a_4 \ldots \succ a_n \ldots$$

This submultitude be denoted by W_1. If the type a_λ be represented by M_λ the following system of expressions is correct:

$$M_2 \mid M_1$$

$$M_3 \mid M_2$$

$$\vdots$$

*) The writer intentionally avoids applying Zermelo's theorem on comparableness of multitudes.

i. e. the sections . . . . M_4 M_3 M_2 are sections of M_1; let m_λ be the element of M_1 which originates M_λ, $\lambda = 2, 3, 4 \ldots$, then

$$m_2 \succ m_3 \succ m_4 \succ \ldots \text{ in infinitum}$$

will follow as M_3 being a section ob M_2 is contained in the latter.

This however is a contradiction, as, M_1 being eutaxitic, the series $m_2 \succ m_3 \succ m_4 \ldots$ must break off behind an element with finite index, therefore a submultitude without a first element does not exist, i. e. W is eutaxitic. The type 1 2 3 4 . . . . or a_1 a_2 a_3 a_4 . . . . is the first element of W, as a prior one would have to be the representative of a multitude homological to a section of a_1 a_2 a_3 a_4 . . . ., i. e. of a finite multitude. The type a_1 a_2 . . . . a_∞ b is the second element of W, as a prior type would be homological to the only infinite section a_1 a_2 . . . . in inf. of a_1 a_2 . . . . b

W possesses no last element: supposing there were a last element, a, corresponding to M, a posterior Multitude M′ is sure to exist, necessitating the existance of an element a′. Considering $M' = (M, x)$, where x does not occur in M, and M′ is to denote the multitude composed of the elements of M with an additional x, M is a section of M′.

W is furthermore not dinumerable, for if it were, that is if

$$W = a_1\ a_2\ a_3\ a_4 \ldots$$

were correct (a_λ denoting the various types), the arrangement not necessarily being eutaxitic, then as

W possesses no last element, a certain ν would follow for which

$$a_\nu \succ a_1$$

takes place. The smallest ν of this kind be characterized by λ_2, so that we get

$$a_1 \prec a_{\lambda_2}$$

$$a_{\lambda_2} \succ \begin{cases} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{\lambda_2 - 1} \end{cases}$$

There is a smallest index λ_3 for which

$$a_{\lambda_3} \succ a_{\lambda_2}$$

is correct, so that

$$1 < \lambda_2 < \lambda_3$$

follows. In general we conclude

$$a_{\lambda_\mu} \succ a_{\lambda_{\mu} - 1}$$

λ_μ again possessing minimum qualities. Thus we arrive at a series of increasing positive intergers:

$$1 < \lambda_2 < \lambda_3 < \lambda_4 \ldots . < \lambda_\nu < \lambda_{\nu+1} \ldots . \text{ in inf.}$$

where a_{λ_ν} is posterior to any element preceding it in the above dinumerable arrangement. Therefore

$$\lambda_\nu \leqq \nu$$

$$\therefore \lambda_{\nu+1} > \nu$$

If a be represented by M_1, a_{λ_2} by M_2, a_{λ_3} by M_3 etc. the following system of sectional relationship is evident:

$$\begin{array}{c|c} M_1 & M_2 \\ M_2 & M_3 \\ \cdot & \\ \cdot & \\ \cdot & \\ \cdot & \end{array}$$

so that R_ν signifying the remainder in question, we are entitled to the following equations:

$$\begin{aligned} M_2 &= M_1 + R_2 \\ M_3 &= M_2 + R_3 \\ &\cdot \\ &\cdot \\ &\cdot \\ &\cdot \\ M_{\nu+1} &= M_\nu + R_{\nu+1} \end{aligned}$$

We now construct a multitude of positive intergers, dinumerable and eutaxitic, (the type of which is not an a_ν) viz:

$$M = M_1 + R_2 + R_3 + R_4 + \ldots \text{ in inf.}$$

If we arrange in such a manner that the elements of M_1 rank first, then those of R_2 etc. the next result

will be a simple arrangement. M consists of parts M_1 R_2 . . . . , one of which, for instance R_4, is the first to contribute elements to a submultitude N of M. These elements for their part form a submultitude of R_4, therefore also of M_4, so that they must possess a first element, which therefore is the first element of N, i. e. M is eutaxitic.

Let a_ν, $\nu < \infty$, denote the type of N, than $a_{\lambda_{\nu+1}}$ is posterior to any a_x, if x is smaller than $\lambda_{\nu+1}$. If ν is the index in question

$$a_{\lambda_{\nu+1}} \succ a_\nu, \text{ as } \lambda_{\nu+1} > \nu$$

follows, so that

$$M \mid M_{\nu+1}$$

were correct, while the definition of M was

$$M_{\nu+1} \mid M$$

wherewith a contradiction against the dinumerability of W is demonstrated.

We will now proceed to prove that the power of W is the lowest but one:

Let M be a submultitude of W, then one only of the following three cases is possible:

$$\begin{array}{llll} 1.) & M \eqsim W & & \\ 2.) & M \eqsim W_1 & \text{where } W_1 & \mid W \\ 3.) & M_1 \eqsim W & \text{"} \quad M_1 & \mid M \end{array}$$

ad 1) $M \risingdotseq W$ includes $M \frown W$ according to the definition of the homological qualities and those of equivalence.

ad 2) $M \risingdotseq W_1$. We must show that each section of W is equivalent either to W or to A. Given a section originated by the type a, the latter representing a dinumerable and eutaxitic multitude N. The section W_1 in consideration consists of all types prior to a, every section of N contributing to the elements of section W_1 of W, and all elements of W_1 corresponding to sections of N, so that the types of W_1 correspond to the sections of N. Therefore we must consider a submultitude of the multitude of sections of N; they are dinumerable in consequence of the monological correlation between themselves and the elements originating them: therefore Malso is dinumerable.

ad 3) $M_1 \risingdotseq W$, If W is homological to a section of M it is also homological, and therefore equivalent to a submultitude of M. As however $M < W$ we find:

$W \frown M'$ M' signifying a submultitude of M
and $M \frown W'$ W' " " " of W
$\therefore$ $M \frown W$ according to the theorem of equivalence
therefore W is of the lowest power but one.

This combined with the result of the last chapter is conclusive evidence for the eutaxy of the continuum.

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C is of the power $\aleph_1$
according to the above W " " " " $\aleph_1$
$\therefore C \frown W$

and as W is a eutaxitic multitude, we conclude that all multitudes equivalent to or comparable with C are capable of eutaxy.

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Vita.

Haroldus Arthurius Penrhynus Pittard - Bullock Sarniae eques Britannicus natus est a. d. VII. Id. Jan. MDCCCLXXXI, matrem Newnhamiae Cantabrigiae valere gaudet, patrem magistrum artium Trinitatis Cantabrigiae, legatum Britannicum regium praemature perdidisse dolet. fidem profitetur Anglicanam. literarum imprimo domi imbutus numero legitime adscriptus civium universitatis Berolinensis per octies sex menses usque ad ver anni domini MCMV literis studiisque operam dedit philosophicis, iuridicis, navalibus technicis nec non mathematicis. scholas audivit virorum illustrium: Flamm, K. Hensel, Hessenberg, E. Landau, Marcuse, Neesen, Riedler, Schollmeyer, H. A. Schwarz, Thiele, de Wilamowitz-Moellendorff, quibus viris omnibus de studiis optime meritis, imprimo prff. doctt. Hensel, Landau, H. A. Schwarz, gratias et nunc agit et semper habebit quam maximas.